PERGAMON

International Journal of Solids and Structures 37 (2000) 1817–1839

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

An asymptotic theory for circular cylindrical shells

Frithiof I. Niordson

Department of Solid Mechanics, Technical University of Denmark, Building 404, DK-2800 Lyngby, Denmark

Received 24 March 1998; in revised form 11 November 1998

Abstract

We derive the two-dimensional shell equations for a circular cylindrical shell by means of an asymptotic expansion of the three-dimensional elastic state. The assumptions involved are of mathematical character only and concern the continuity, differentiability and convergence of the series used. A numerical comparison between the frequencies obtained by the two-dimensional shell equations and the three-dimensional state in the case of free vibrations is presented in a few examples.

1. Introduction

Elastic shell theories are usually derived from the equations of elasticity augmented by one or more assumptions concerning the state of stress, deformation or elastic energy. Usually the 'Kirchhoff hypothesis' is invoked in one way or another (Simmonds, 1997). In this paper the two-dimensional theory of thin elastic circular cylindrical shells is derived solely from the three-dimensional theory of elasticity using the method of asymptotic expansion. Any assumption made is of purely mathematical character concerning continuity, differentiability and convergence of series. There is no 'Kirchhoff hypothesis' or any other hypotheses invoked, either in the constitutive relations or elsewhere. It is an exact theory. Many might find the theory overly fancy and the mathematics unnecessarily elaborate, but that is how things are, it cannot be otherwise. I do not claim that any refined shell theory for cylindrical shells is strongly needed, but I do claim that a linear shell theory built on First Principles only, is something new and has an immediate appeal.

The method of asymptotic expansion of the exact three-dimensional linear theory of elasticity has earlier been applied to thin plates (Brod, 1972; Niordson, 1979), and in this paper an extension to circular cylindrical shells is presented. Following that method, we expand all quantities in terms of a small parameter h/L, where $2h$ is the thickness of the shell and L is a characteristic length of the deformation pattern and obtain a sequence of shell equations, each more accurate than the preceding

E-mail address: famfn@pop.dtu.dk (F.I. Niordson)

0020-7683/00/$ - see front matter
PII: S0020-7683(98)00331-X

one. Here L can be taken to be the radius R of the shell, or as in the case of free vibrations, the wavelength c/ω, where c is the velocity of sound in the material and ω the angular frequency.

With this method we obtain the equations of equilibrium as a power series in h, where the coefficients are linear differential operators on the displacement functions. The order of the theory will be denoted by n, the highest power at which the series is truncated.

The zero-order equations are identically satisfied, and there is therefore no zero-order terms and no zero-order theory. In the first-order theory ($n=1$), after cancelling a common factor, we find the well-known membrane theory of cylindrical shells, in which the solution is independent of the thickness of the shell.

The second-order theory ($n=2$) gives an unexpected result: there are no terms of order h in the asymptotic expansion. The result is certainly not trivial, since it has been generally accepted (Novoshilov and Finkel'shtein, 1943; Koiter, 1960; Niordson, 1971) that by using the uncoupled Love–Kirchhoff strain energy expression, errors of order $|h/R|$ are introduced, where R is the (smallest) radius of principal curvature. Now we find that this is not the case for cylindrical shells.[1] Furthermore, by careful analysis, we find that, like in the case of flat plates, only odd numbers of n contribute to the asymptotic expansion. For flat plates, this was an obvious consequence of the symmetry properties in normal direction. For cylindrical shells, where no such symmetry is present, it follows from the analysis.

The following iteration ($n=3$) contains terms of order h^2, and is therefore the lowest order bending theory of cylindrical shells. It does not coincide with any earlier proposed bending theory, as far as the author is aware, but the difference from, for example, the Morley–Koiter equations (Morley, 1959; Koiter, 1968) seems to be insignificant, at least numerically.[2]

With the formulas and procedures derived in this paper we can proceed to obtain any higher-order theory. Thus the fifth order theory ($n=5$) is also derived and presented here. It includes terms of order h^4 and in the numerical examples this theory shows a considerable improvement of accuracy over the classical shell theory for sufficiently thin shells. But this is as far as we go. Higher order theories are not derived in this paper, if only for the fact that the equations and formulas become rather awkward to handle. Also the order of the differential equations increases. This was also found to be the case for plates, however in that case, with a single dependent variable (the normal displacement) and a single differential operator (the Laplacian), it was possible to find a way by which the infinite series could be summed, hereby reducing the order of the final equation to a fourth-order differential equation, the same order as the Kirchhoff equation. For cylindrical shells with three dependent variables (the displacement components) and two differential operators (the partial derivatives in the middle surface), unfortunately, no such reduction has been found.

The method derived can be applied to statically loaded shells as well as to shells in free vibrations, but the main part of the analysis is devoted to the dynamic case of free vibrations.

The method of asymptotic expansion yields an exact solution (provided the series converge), which in the limit coincides with a solution of the three-dimensional problem. But the truncated asymptotic series is of course only an approximation. To illustrate the accuracy of the two-dimensional shell theories, some numerical results have been obtained for the third- and fifth-order theory and compared with a particular (numerical) solution to the three-dimensional equations of motion of an infinitely long cylinder in free vibrations.

These examples show that the 'refined' fifth-order theory is far superior to the classical third-order theory for sufficiently thin shells. However, they also show very clearly that shell theories are limited to thin shells.

[1] We conjecture that the same holds true for all shells.
[2] Niordson, F.I., 1985, p. 229.

2. Basic equations

Let $x^i = (x, y, z)$ be normal coordinates,[3] where z is the distance from the middle surface, $z = +h$ being the outer surface and $z = -h$ the inner surface of the shell. The coordinates x and y on the middle surface are rectangular, x parallel to the axis of the cylinder and y the arc-length in tangential direction. The boundary will be defined by the normals to the middle surface along one or two simple closed curves C on the middle surface.

We shall furthermore assume that the shell performs small harmonic vibrations of amplitude $u^i = (u, v, w)$ and angular frequency ω. Through this frequency we define a wavelength $L = c/\omega$, where c is the velocity of sound in the material, which is supposed to be homogeneous, isotropic and linearly elastic, following Hooke's law.

In the normal coordinate system the covariant metric tensor g_{ij} has the components[4]

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1 + z/R)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the corresponding contravariant components are

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1 + z/R)^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where R is the radius of the middle surface.

Also the Christoffel symbols $\left\{ {i \atop jk} \right\}$ are functions of z, however, we find that all vanish, except the following ones

$$\left\{ {3 \atop 22} \right\} = -\frac{1 + z/R}{R}$$

$$\left\{ {2 \atop 23} \right\} = \left\{ {2 \atop 32} \right\} = \frac{1/R}{1 + z/R}$$

For given conditions at the boundary the amplitude functions will depend on the thickness of the shell element, and this dependence will be represented by the following asymptotic expansion,

$$u^i(x, y, z) = \sum_{n=0}^{\infty} u^i_{(n)}(x, y, z)\varepsilon^n$$

for the contravariant components. Here the dimensionless number

[3] Latin indices are used for the range 1, 2, 3.

[4] Niordson, F.I., 1985, p. 48.

$$\varepsilon = h/L \tag{1}$$

is assumed to be small in comparison with unity.

The functions $u^i_{(n)}(x, y, z)$ are now expanded in Taylor series at $z = 0$, i.e.

$$u^i(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} U^i_{(n,m)}(x, y) z^m \varepsilon^n$$

or

$$u^i = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} U^i_{(n,m)} z^m \varepsilon^n$$

where

$$U^i_{(n,m)} = \frac{\partial^m u^i_{(n)}}{\partial z^m}(x, y, 0) \quad n, m = 0, 1 \ldots$$

are the partial derivatives of $u^i_{(n)}$ with respect to z at the middle-surface. It follows that the displacements of the middle-surface, with which the two-dimensional theory shall deal, are given by

$$u^i(x, y, 0) = \sum_{n=0}^{\infty} U^i_{(n,0)} \varepsilon^n$$

In the following, whenever convenient, we shall use u, v, w to denote u^i and U, V, W to denote U^i.

The stress tensor $\sigma^{ij}(x, y, z)$ is given in terms of the displacements by Hooke's law,

$$\sigma^{ij} = G\left(\mathscr{D}^i u^j + \mathscr{D}^j u^i + \mu g^{ij} \mathscr{D}_k u^k\right)$$

where G is the shear modulus. The script letter $\mathscr{D}$ denotes the covariant derivative (or contravariant derivative—as the case may be) in three dimensions.

We have also used the shorter notation for the following combination

$$\mu = \frac{2\nu}{1 - 2\nu}$$

of Poisson's ratio ν.

Writing the stresses in dimensionless form

$$\Sigma^{ij} = \mathscr{D}^i u^j + \mathscr{D}^j u^i + \mu g^{ij} \mathscr{D}_k u^k \tag{2}$$

we have only one material constant (μ) left in our formulas. Note that for $\nu = 1/3$ the number $\mu = 2$.

The stresses are expanded in the same way as the displacements,

$$\Sigma^{ij}(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} s^{ij}_{(n,m)}(x, y) z^m \varepsilon^n \tag{3}$$

The equations of motion are

$$\mathscr{D}_i \Sigma^{ij} + \Lambda u^j = 0 \tag{4}$$

where

$$\Lambda = \rho\omega^2/G$$

The dependence of Λ on ε is given by

$$\Lambda = \sum_{n=0}^{\infty} \Lambda_{(n)}\varepsilon^n$$

Using (2) and (4) the equations of motion may be expressed in terms of the displacements

$$\mathscr{D}_i\mathscr{D}^i u^j + \mathscr{D}_i\mathscr{D}^j u^i + \mu\mathscr{D}^j\mathscr{D}_k u^k + \Lambda u^j = 0$$

Expanding the covariant derivatives, we get the following three equations

$$\frac{\partial^2 u}{\partial y^2} + \frac{1}{R}\left(1+\frac{z}{R}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) + \left(1+\frac{z}{R}\right)^2\left(2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial z}+\frac{\partial^2 u}{\partial z^2}\right)$$
$$+\mu\left(1+\frac{z}{R}\right)\left[\frac{1}{R}\frac{\partial w}{\partial x}+\left(1+\frac{z}{R}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial z}\right)\right]+\Lambda\left(1+\frac{z}{R}\right)^2 u = 0 \tag{5}$$

$$\frac{3}{R}\frac{\partial w}{\partial y} + \left(1+\frac{z}{R}\right)\left(2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial y\,\partial z}\right)+\frac{3}{R}\left(1+\frac{z}{R}\right)^2\frac{\partial v}{\partial z}+\left(1+\frac{z}{R}\right)^3\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}\right)$$
$$+\mu\left[\frac{1}{R}\frac{\partial w}{\partial y}+\left(1+\frac{z}{R}\right)\left(\frac{\partial^2 w}{\partial y\,\partial z}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial x\,\partial y}\right)\right]+\Lambda\left(1+\frac{z}{R}\right)^3 v = 0 \tag{6}$$

and

$$\frac{\partial^2 w}{\partial y^2} - 2\frac{w}{R^2} + \frac{2}{R}\left(1+\frac{z}{R}\right)\left(\frac{\partial w}{\partial z}+\frac{\partial v}{\partial y}\right)+\left(1+\frac{z}{R}\right)^2\left(2\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 v}{\partial y\,\partial z}+\frac{\partial^2 u}{\partial x\,\partial z}+\frac{\partial^2 w}{\partial x^2}\right)$$
$$+\mu\left[-\frac{w}{R^2}+\frac{1}{R}\left(1+\frac{z}{R}\right)\frac{\partial w}{\partial z}+\left(1+\frac{z}{R}\right)^2\left(\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 v}{\partial y\,\partial z}+\frac{\partial^2 u}{\partial x\,\partial z}\right)\right]+\Lambda\left(1+\frac{z}{R}\right)^2 w = 0 \tag{7}$$

Substituting the displacements in the equations of motion, and equating the coefficients of all powers of z and ε to zero, the equations of motion (5)–(7) generate the following relations between the derivatives,

$$\begin{aligned}
&(2+\mu)\frac{m(m-1)}{R^2}\delta_x^2 U_{(n,m-2)} + 2(2+\mu)\frac{m}{R}\delta_x^2 U_{(n,m-1)} + \left[(2+\mu)\delta_x^2+\delta_y^2+\frac{m^2}{R^2}\right]U_{(n,m)}\\
&+\frac{1+2m}{R}U_{(n,m+1)} + U_{(n,m+2)} + (1+\mu)\left(\frac{m(m-1)}{R^2}\delta_x\delta_y V_{(n,m-2)} + 2\frac{m}{R}\delta_x\delta_y V_{(n,m-1)} + \delta_x\delta_y V_{(n,m)}\right.\\
&\left.+\frac{m^2}{R^2}\delta_x W_{(n,m-1)} + \frac{1+2m}{R}\delta_x W_{(n,m)} + \delta_x W_{(n,m+1)}\right) + \sum_{r=0}^{n}\Lambda_{(r)}\left[\frac{m(m-1)}{R^2}U_{(n-r,m-2)}\right.\\
&\left.+2\frac{m}{R}U_{(n-r,m-1)} + U_{(n-r,m)}\right] = 0
\end{aligned} \tag{8}$$

$$
\begin{aligned}
&(1+\mu)\left[\frac{m}{R}\delta_x\partial_y U_{(n,m-1)}+\delta_x\partial_y U_{(n,m)}\right]+\frac{m(m-1)(m-2)}{R^3}\delta_x^2 V_{(n,m-3)}+3\frac{m(m-1)}{R^2}\delta_x^2 V_{(n,m-2)}\\
&+\frac{m}{R}\left[3\delta_x^2+(2+\mu)\delta_y^2-\frac{m^2-1}{R^2}\right]V_{(n,m-1)}+\left[\delta_x^2+(2+\mu)\delta_y^2+3\frac{m(m+1)}{R^2}\right]V_{(n,m)}\\
&+3\frac{m+1}{R}V_{(n,m+1)}+V_{(n,m+2)}+\frac{m+3+\mu(m+1)}{R}\delta_y W_{(n,m)}+(1+\mu)\delta_y W_{(n,m+1)}\\
&+\sum_{r=0}^{n}\Lambda_{(r)}[\frac{m(m-1)(m-2)}{R^3}V_{(n-r,m-3)}-\frac{3m(m-1)}{R^2}V_{(n-r,m-2)}+3\frac{m}{R}V_{(n-r,m-1)}+V_{(n-r,m)}]=0
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&(1+\mu)\left[\frac{m(m-1)}{R^2}\delta_x U_{(n,m-1)}+2\frac{m}{R}\delta_x U_{(n,m)}+\delta_x U_{(n,m+1)}\right]+\frac{m(1+\mu)-3-\mu}{R^2}\delta_y V_{(n,m-1)}\\
&+2\frac{m(\mu+1)-1}{R}\delta_y V_{(n,m)}+(1+\mu)\delta_y V_{(n,m+1)}+\frac{m(m-1)}{R^2}\delta_x^2 W_{(n,m-2)}+2\frac{m}{R}\delta_x^2 W_{(n,m-1)}\\
&+\left[\delta_x^2+\delta_y^2+(2+\mu)\ \frac{m^2-1}{R^2}\right]W_{(n,m)}+(2+\mu)\left[\frac{1+2m}{R}W_{(n,m+1)}+W_{(n,m+2)}\right]\\
&+\sum_{r=0}^{n}\Lambda_{(r)}\left[\frac{m(m-1)}{R^2}W_{(n-r,m-2).}+2\frac{m}{R}W_{(n-r,m-1)}+W_{(n-r,m)}\right]=0
\end{aligned}
\tag{10}
$$

respectively. Here δ_x and δ_y are the differential operators $\partial/\partial x$ and $\partial/\partial y$, respectively.

From Hooke's law (2) and the expansion (3) the stress components are found to be

$$
\begin{aligned}
&S^{11}_{(n,m)}+\frac{m}{R}S^{11}_{(n,m-1)}=(2+\mu)\frac{m}{R}\delta_x U_{(n,m-1)}+(2+\mu)\delta_x U_{(n,m)}+\mu\frac{m}{R}\delta_y V_{(n,m-1)}+\mu\delta_y V_{(n,m)}\\
&+\mu\frac{m+1}{R}W_{(n,m)}+\mu W_{(n,m+1)}
\end{aligned}
$$

$$
S^{12}_{(n,m)}+2\frac{m}{R}S^{12}_{(n,m-1)}+\frac{m(m-1)}{R^2}S^{12}_{(n,m-2)}=\delta_y U_{(n,m)}+\frac{m(m-1)}{R^2}\delta_x V_{(n,m-2)}+2\frac{m}{R}\delta_x V_{(n,m-1)}+\delta_x V_{(n,m)}
$$

$$
\begin{aligned}
&S^{22}_{(n,m)}+3\frac{m}{R}S^{22}_{(n,m-1)}+3\frac{m(m-1)}{R^2}S^{22}_{(n,m-2)}+\frac{m(m-1)(m-2)}{R^3}S^{22}_{(n,m-3)}=\mu\frac{m}{R}\delta_x U_{(n,m-1)}\\
&+\mu\delta_x U_{(n,m)}+(2+\mu)\left[\frac{m}{R}\delta_y V_{(n,m-1)}+(2+\mu)\delta_y V_{(n,m)}\right]+\frac{\mu(m+1)+2}{R}W_{(n,m)}+\mu W_{(n,m+1)}
\end{aligned}
$$

$$
S^{31}_{(n,m)}=U_{(n,m+1)}+\delta_x W_{(n,m)}
\tag{11}
$$

$$
S^{32}_{(n,m)}+2\frac{m}{R}S^{32}_{(n,m-1)}+\frac{m(m-1)}{R^2}S^{32}_{(n,m-2)}=\frac{m(m-1)}{R^2}V_{(n,m-1)}+2\frac{m}{R}V_{(n,m)}+V_{(n,m+1)}+\delta_y W_{(n,m)}
\tag{12}
$$

$$S^{33}_{(n,m)} + \frac{m}{R} S^{33}_{(n,m-1)} = \mu \frac{m}{R} \delta_x U_{(n,m-1)} + \mu \delta_x U_{(n,m)} + \mu \frac{m}{R} \delta_y V_{(n,m-1)} + \mu \delta_y V_{(n,m)} + \frac{m(2+\mu)+\mu}{R} W_{(n,m)} + (2+\mu) W_{(n,m+1)} \tag{13}$$

The boundary conditions at the free surfaces $z = \pm h$ are given by

$$\Sigma^{i3}(x, y, +h) = P^i_+(x, y)$$

$$\Sigma^{i3}(x, y, -h) = P^i_-(x, y) \tag{14}$$

where P^i_+ and P^i_- are the external loads on the outer and inner surface of the cylinder, respectively.

The static problem of an externally loaded cylindrical shell requires that we take $\Lambda = 0$ and give the external forces P^i_+ and P^i_- on the outer and inner surface of the cylinder as functions of x and y.

For the dynamic problem of free vibrations, we keep Λ but take the external forces P^i_+ and P^i_- to be equal to zero.

The further treatment of both cases is quite similar. The static problem leads to a system of three inhomogeneous differential equations with a unique solution. The dynamic one leads to a system of three homogeneous differential equations, i.e. an eigenvalue problem. In both cases the main effort lies in determining the coefficients of the displacement functions and their determinant.

To avoid a tedious repetition we shall confine our further analysis to the case of free vibrations.

Now, assuming that the external forces on the outer and inner surfaces vanish, we have

$$\sum_{r=0}^{n} \frac{(\pm 1)^{n-r}}{(n-r)!} S^{i3}_{(n-r,r)} L^{n-r} = 0 \quad i = 1, 2, 3 \quad n = 0, 1, \dots \tag{15}$$

These are six boundary conditions for each value of n, three for the outer surface (the plus sign) and three for the inner surface (the minus sign).

The basic equations, from which the two-dimensional shell equations can be obtained, are now derived. In the next section, we shall proceed by eliminating the unknown derivatives.

3. Elimination of the derivatives

In order to obtain the two-dimensional equations for the shell, we must eliminate all derivatives with respect to z by expressing $U^i_{(n,m)}$ for $m > 0$ in terms of $U^i_{(r,0)}$ for $r = 0, 1, 2, \dots, n$.

Consider the following matrix,

$$\begin{bmatrix} U^i_{(0,0)} & U^i_{(0,1)} & U^i_{(0,2)} & \cdots \\ U^i_{(1,0)} & U^i_{(1,1)} & U^i_{(1,2)} & \cdots \\ U^i_{(2,0)} & U^i_{(2,1)} & U^i_{(2,2)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

The first column contains the 'given' functions, in terms of which all the remaining functions are to be expressed. By following the 'slash-order' indicated by the Italian numbers in the diagram below, we can determine any derivative in terms of zero-order derivatives, since it will depend only on earlier derived functions, all of which are already expressed in terms of the zero-order derivatives ($m = 0$).

$U^i_{(n,m)}$

n\m	1	2	3	4
0	*1*	*2*	*4*	*7*
1	*3*	*5*	*8*	
2	*6*	*9*		
3	*10*			

The elements in the second column of the matrix are the first derivatives of the displacement functions. They are found from the boundary conditions (15) in the following way. Since

$$\Sigma^{i3}(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} S^{i3}_{(n,m)} z^m \varepsilon^n \quad i = 1, 2, 3$$

we get at the outer and inner surface of the cylinder

$$\Sigma^{i3}(x, y, \pm h) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} S^{i3}_{(n,m)} (\pm L)^m \varepsilon^m \varepsilon^n = 0 \quad i = 1, 2, 3$$

which holds true if

$$\sum_{r=0}^{n} \frac{(\pm L)^{n-r}}{(n-r)!} S^{i3}_{(r,n-r)} = 0 \quad i = 1, 2, 3 \quad n = 0, 1, 2 \ldots$$

This can be written

$$\sum_{r=0}^{n-1} \frac{(\pm L)^{n-r}}{(n-r)!} S^{i3}_{(r,n-r)} + S^{i3}_{(n,0)} = 0 \quad i = 1, 2, 3 \quad n = 0, 1, 2 \ldots$$

The condition that *the sum of the stresses* on the outer and inner boundaries is zero, will therefore be

$$\sum_{r=0}^{n-1} \frac{L^{n-r} + (-L)^{n-r}}{(n-r)!} S^{i3}_{(r,n-r)} + 2S^{i3}_{(n,0)} = 0$$

With the help of the eqns (11)–(13) for $m=0$ we find the elements of the second column from the formulas

$$U_{(n,1)} = -\delta_x W_{(n,0)} + \frac{1}{2} \sum_{r=0}^{n-1} \frac{L^{n-r} + (-L)^{n-r}}{(n-r)!} S^{31}_{(r,n-r)}$$

$$V_{(n,1)} = -\delta_y W_{(n,0)} + \frac{1}{2} \sum_{r=0}^{n-1} \frac{L^{n-r} + (-L)^{n-r}}{(n-r)!} S^{32}_{(r,n-r)}$$

$$W_{(n,1)} = \left[-\mu\left(\delta_x U_{(n,0)} + \delta_y V_{(n,0)}\right) - \frac{1}{R} W_{(n,0)} + \frac{1}{2} \sum_{r=0}^{n-1} \frac{L^{n-r} + (-L)^{n-r}}{(n-r)!} S^{33}_{(r,n-r)} \right] \Big/ (2 + \mu)$$

By this choice we have ensured that for each component the sum of the stresses on the outer and inner surfaces vanishes. Later, in deriving the final equations, we shall impose the condition that also the difference does vanish, thus making sure that the stresses on both sides are zero.

The derivatives of order $m > 1$, i.e. all elements in the third, fourth and higher number columns can be found by solving the equations of equilibrium (8)–(10) for the highest-order derivative, thus obtaining the derivatives of order $m+2$ from the lower-order derivatives.

To make the eliminations procedure clear, we shall give the first steps in detail. Below, the numbers in square brackets correspond to the (Italic) numbers in the table.

[1] The first derivatives with respect to z of the zero-order functions $U^i_{(0,1)}$ are found from the boundary conditions (15), which for $n=0$ reduce to $S^{i3}_{(0,0)} = 0$. With $n=m=0$ the eqns (11)–(13) yield

$$U_{(0,1)} = -\delta_x W_{(0,\,0)}$$

$$V_{(0,1)} = -\delta_y W_{(0,\,0)}$$

$$W_{(0,1)} = -\frac{1}{2+\mu}\left(\mu\delta_x U_{(0,0)} + \mu\delta_y W_{(0,0)} + \frac{\mu}{R} W_{(0,0)}\right)$$

[2] The second derivative $U_{(0,2)}$ is found from the equation of motion (8), which for $n=m=0$ reduces to

$$U_{(0,2)} = -\frac{4\delta_x^2 + 2\delta_y^2 + 3\mu\delta_x^2 + \mu\delta_y^2}{2+\mu} U_{(0,0)} - 2\frac{1+\mu}{2+\mu}\delta_x\delta_y V_{(0,0)} - \frac{\mu}{R(2+\mu)}\delta_x W_{(0,0)} - \Lambda_{(0)} U_{(0,0)}$$

and similarly, $V_{(0,2)}$, $W_{(0,2)}$ are found from eqns (9) and (10).

[3] The first derivatives with respect to z of the first-order functions $U^i_{(1,1)}$ are again found using the boundary conditions (15), which for $n = 1$ yield $S^{i3}_{(1,0)} = 0$. With $n=1$ and $m=0$ the eqns (11)–(13) yield

$$U_{(1,1)} = -\delta_x W(1,0)$$

$$V_{(1,1)} = -\delta_y W(1,0)$$

$$W_{(1,1)} = -\frac{1}{2+\mu}\left(\mu\delta_x U_{(1,0)} - \mu\delta_y W_{(1,0)} - \frac{\mu}{R} W_{(1,0)}\right)$$

The six derivatives of [2] and [3] completes the first ‘slash’. This is all, what is needed for the lowest-order theory. The derivatives in the next ‘slash’ consisting of [4], [5], [6] are determined correspondingly, and so on.

When the relevant derivatives have been deduced, the boundary conditions can be applied to find the differential equations for the shell. Now we impose the restriction that the difference of the stresses between the outer and inner surface is zero. Their structure reveals that for each n they impose one relation between the functions $U_{(n-r,r)}$, $V_{(n-r,r)}$ and $W_{(n-r,r)}$ for $r = 0, 1, 2, \ldots, n$, indicating the economy of the computational work of the ‘slash-order’ sequence.

For $n = 0$ these relations are identically satisfied, actually because they were used to derive the functions $U_{(0,1)}$, $V_{(0,1)}$ and $W_{(0,1)}$. However, for all $n > 0$ they given non-trivial results.

4. The sequence of asymptotic equations

For $n = 1$ we get the first non-trivial approximation to the equations of cylindrical shells. After determining the first nine derivatives

$$U_{(0,1)}, \quad V_{(0,1)}, \quad W_{(0,1)}, \quad U_{(0,2)}, \quad V_{(0,2)}, \quad W_{(0,2)}, \quad U_{(1,1)}, \quad V_{(1,1)}, \quad W_{(1,1)}$$

which were deduced using the condition that the sum of the stresses at the outer and inner boundaries did vanish, we apply the boundary conditions (15) again, but not to the difference between the stresses at the outer and inner surface.

$$\sum_{r=0}^{n} \frac{L^{n-r} - (-L)^{n-r}}{(n-r)!} S^{i3}_{(n-r,r)} = 0 \quad i = 1, 2, 3$$

This yields the following three equations

$$\begin{aligned} &F_1\left[U_{(0,0)}, \quad V_{(0,0)}, \quad W_{(0,0)}\right] + \Lambda_{(0)} U_{(0,0)} = 0 \\ &F_2\left[U_{(0,0)}, \quad V_{(0,0)}, \quad W_{(0,0)}\right] + \Lambda_{(0)} V_{(0,0)} = 0 \\ &F_3\left[U_{(0,0)}, \quad V_{(0,0)}, \quad W_{(0,0)}\right] + \Lambda_{(0)} W_{(0,0)} = 0 \end{aligned} \tag{16}$$

where the linear differential operators F_1, F_2 and F_3 are given by

$$F_i[u, v, w] = \frac{1}{2+\mu} \sum_p \sum_q \left(\frac{\partial}{\partial x}\right)^p \left(\frac{\partial}{\partial y}\right)^q f_{ipq}(x, y) \quad i = 1, 2, 3$$

in which the functions $f_{ipq}(x, y)$ are given in Appendix I.

These are the shell equations for the membrane state, and it is easily checked that they correspond precisely to the classical shell equations taken in the limit $h/R = 0$, i.e. when no bending terms are present.

Proceeding in the way described in the last section, we get for $n = 2$, after determining the nine additional derivatives,

$$U_{(0,3)}, \quad V_{(0,3)}, \quad W_{(0,3)}, \quad U_{(1,2)}, \quad V_{(1,2)}, \quad W_{(1,2)}, \quad U_{(2,1)}, \quad V_{(2,1)}, \quad W_{(2,1)}$$

the equations

$$\begin{aligned} &F_1\left[U_{(1,0)}, V_{(1,0)}, W_{(1,0)}\right] + \Lambda_{(0)} U_{(1,0)} + \Lambda_{(1)} U_{(0,0)} = 0 \\ &F_2\left[U_{(1,0)}, V_{(1,0)}, W_{(1,0)}\right] + \Lambda_{(0)} V_{(1,0)} + \Lambda_{(1)} V_{(0,0)} = 0 \\ &F_3\left[U_{(1,0)}, V_{(1,0)}, W_{(1,0)}\right] + \Lambda_{(0)} W_{(1,0)} + \Lambda_{(1)} W_{(0,0)} = 0 \end{aligned} \tag{17}$$

with the same linear operators F_1, F_2 and F_3 as before. Comparison of eqns (16) and (17) shows that $\Lambda_{(1)} = 0$. It is therefore clear that this iteration adds nothing of essence to the theory. In fact, as in the case of flat plates, only odd numbered iterations provide non-trivial results.

For $n = 3$ we need in addition the following 12 derivatives,

$$U_{(0,4)}, \quad V_{(0,4)}, \quad W_{(0,4)}, \quad U_{(1,3)}, \quad V_{(1,3)}, \quad W_{(1,3)}$$

$$U_{(2,2)}, \quad V_{(2,2)}, \quad W_{(2,2)}, \quad U_{(3,1)}, \quad V_{(3,1)}, \quad W_{(3,1)}$$

and obtain the third-order equations of equilibrium

$$
\begin{aligned}
&F_1\big[U_{(2,0)}, V_{(2,0)}, W_{(2,0)}\big] + L^2G_1\big[U_{(0,0)}, V_{(0,0)}, W_{(0,0)}\big] + \Lambda_{(0)}U_{(2,0)} + \Lambda_{(2)}U_{(0,0)} = 0 \\
&F_2\big[U_{(2,0)}, V_{(2,0)}, W_{(2,0)}\big] + L^2G_2\big[U_{(0,0)}, V_{(0,0)}, W_{(0,0)}\big] + \Lambda_{(0)}V_{(2,0)} + \Lambda_{(2)}V_{(0,0)} = 0 \\
&F_3\big[U_{(2,0)}, V_{(2,\,0)}, W_{(2,0)}\big] + L^2G_3\big[U_{(0,\,0)}, V_{(0,0)}, W_{(0,0)}\big] + \Lambda_{(0)}W_{(2,0)} + \Lambda_{(2)}W_{(0,0)} = 0
\end{aligned} \tag{18}
$$

where the linear differential operators G_1, G_2, and G_2 are given by

$$
G_i[u, v, w] = \frac{1}{3(2+\mu)^3}\sum_p\sum_q\left(\frac{\partial}{\partial x}\right)^p\left(\frac{\partial}{\partial y}\right)^q g_{ipq}(x, y)
$$

in which the functions $g_{ipq}(x, y)$ are given in Appendix I.

In the same way as we could conclude from the second-order iteration that $\Lambda_{(1)} = 0$, we find that the fourth-order iteration implies that $\Lambda_{(3)} = 0$. Next non-trivial iteration occurs for $n = 5$. We find after computing all relevant derivatives that

$$
\begin{aligned}
&F_1\big[U_{(4,0)}, V_{(4,0)}, W_{(4,0)}\big] + L^2G_1\big[U_{(2,0)}, V_{(2,0)}, W_{(2,0)}\big] + L^4H_1\big[U_{(0,0)}, V_{(0,0)}, W_{(0,0)}\big] \\
&\quad + \Lambda_{(0)}U_{(4,0)} + \Lambda_{(2)}U_{(2,0)} + \Lambda_{(4)}U_{(0,0)} = 0 \\
&F_2\big[U_{(4,0)}, V_{(4,0)}, W_{(4,0)}\big] + L^2G_2\big[U_{(2,0)}, V_{(2,0)}, W_{(2,0)}\big] + L^4H_2\big[U_{(0,0)}, V_{(0,0)}, W_{(0,0)}\big] \\
&\quad + \Lambda_{(0)}U_{(4,0)} + \Lambda_{(2)}V_{(2,0)} + \Lambda_{(4)}V_{(0,0)} = 0 \\
&F_3\big[U_{(4,0)}, V_{(4,0)}, W_{(4,0)}\big] + L^2G_3\big[U_{(2,0)}, V_{(2,0)}, W_{(2,0)}\big] + L^4H_3\big[U_{(0,0)}, V_{(0,0)}, W_{(0,0)}\big] \\
&\quad + \Lambda_{(0)}\,W_{(4,0)} + \Lambda_{(2)}\,W_{(2,0)} + \Lambda_{(4)}\,W_{(0,0)} = 0
\end{aligned} \tag{19}
$$

where the linear differential operators H_1, H_2, and H_3 are given by

$$
H_i[u, v, w] = \frac{1}{45(2+\mu)^5}\sum_p\sum_q\left(\frac{\partial}{\partial x}\right)^p\left(\frac{\partial}{\partial y}\right)^q h_{ipq}(x, y)
$$

in which the functions $h_{ipq}(x, y)$ are given in Appendix I.

Multiplying the eqn (18) by ε^2, eqn (19) by ε^4, etc. and adding all these equations to eqn (16), we get

$$
F_1[u, v, w] + h^2G_1[u, v, w] + h^4H_1[u, v, w] + \Lambda u \cdots = 0 \tag{20}
$$

where the dots indicate higher-order terms in the thickness h, and where the displacement functions u, v, w are evaluated at the middle surface, i.e.

$$
u = u(x, y, 0) \quad v = v(x,\ y, 0) \quad w = w(x,\ y, 0)
$$

Similarly, we obtain the second and third equation of motion in the form

$$
F_2[u, v, w] + h^2G_2[u, v, w] + h^4H_2[u, v, w] + \Lambda v \cdots = 0 \tag{21}
$$

$$F_3[u, v, w] + h^2 G_3[u, v, w] + h^4 H_3[u, v, w] + \Lambda w \cdots = 0 \tag{22}$$

where again, the dots indicate higher-order terms in the thickness.

The system of eqns (20)–(22) are the two-dimensional equations for cylindrical shells truncated at h^4 with a relative error of order $(h/L)^6$. Next approximation would require two more iterations, but we stop here. One reason is that the equations grow in length with a factor of roughly 10 for each second iteration and become rather unmanageable.

Written out, the three equations of motion for the lowest order bending theory of shells are

$$\begin{aligned}
&\delta_x^2 u + \frac{1-\nu}{2}\delta_y^2 + \frac{1+\nu}{2}\delta_x\delta_y v + \frac{\nu}{R}\delta_x w + \frac{h^2}{6(1-\nu)^2}\Bigg[2\nu^2\delta_x^4 u + 2\nu^2\delta_x^2\delta_y^2 u + \frac{(1-\nu)^3}{R^2}\delta_y^2 u \\
&\quad - \nu\frac{2-9\nu+6\nu^2}{R^2}\delta_x^2 u + 2\nu^2\delta_x^3\delta_y v + 2\nu^2\delta_x\delta_y^3 v - \frac{(2-\nu)\big(1-3\nu+\nu^2\big)}{R^2}\delta_x\delta_y v - \frac{2-9\nu+6\nu^2}{R^3}\delta_x w \\
&\quad - \frac{2-5\nu+\nu^2}{R}\delta_x^3 w + \frac{1-\nu+4\nu^2-2\nu^3}{R}\delta_x\partial_y^2 w\Bigg] + \frac{1-\nu}{2}\Lambda u = 0
\end{aligned}$$

$$\begin{aligned}
&\frac{1+\nu}{2}\delta_x\delta_y u + \delta_y^2 v + \frac{1-\nu}{2}\delta_x^2 v + \frac{1}{R}\delta_y w + \frac{h^2}{6(1-\nu)^2}\Bigg[2\nu^2\delta_x^3\delta_y u + 2\nu^2\delta_x\delta_y^3 u - \frac{3+\nu-17\nu^2+12\nu^3}{R^2}\delta_x\delta_y u \\
&\quad + 2\nu^2\delta_x^2\delta_y^2 v + 2\nu^2\delta_y^4 v - \frac{(2-3\nu)(5-4\nu)}{R^2}\delta_y^2 v - \frac{10-26\nu+15\nu^2}{R^3}\delta_y w + \frac{\nu(3-\nu)}{R}\delta_y^3 w \\
&\quad + \frac{-3+9\nu-6\nu^2+2\nu^3}{R}\delta_x^2\delta_y w\Bigg] + \frac{1-\nu}{2}\Lambda v = 0
\end{aligned}$$

$$\begin{aligned}
&\frac{\nu}{R}\delta_x u + \frac{1}{R}\delta_y v + \frac{w}{R^2} + \frac{h^2}{6(1-\nu)^2}\Bigg[2\frac{\nu(1-\nu)(1+3\nu)}{R^3}\delta_x u + 3\frac{\nu^2}{R}\delta_x^3 u + \frac{4-5\nu+\nu^2+3\nu^3}{R}\delta_x\delta_y^2 u \\
&\quad + 5\frac{\nu(1-\nu)}{R^3}\delta_y v + 3\frac{2-4\nu+3\nu^2}{R}\delta_y^3 v + \frac{2-7\nu+11\nu^2-3\nu^3}{R}\delta_x^2\delta_y v + 2(1-\nu)^2\Big(\delta_x^4 w + 2\delta_x^2\delta_y^2 w + \delta_y^4 w\Big) \\
&\quad + 2\frac{(1-\nu)(1+3\nu)}{R^4} w + \frac{10-17\nu+10\nu^2}{R^2}\delta_y^2 w + \frac{2-2\nu+\nu^2+2\nu^3}{R^2}\delta_x^2 w\Bigg] - \frac{1-\nu}{2}\Lambda w = 0
\end{aligned}$$

where Poisson's ratio ν has been reinstated for a more convenient comparison with the shell equations of other authors.

5. Boundary conditions

The conditions at the free surfaces $\pm h$ are already taken care of when deriving the asymptotic equations of motion. It remains to formulate the conditions along the boundary curve, in case the shell is not complete.

To have a well-posed mathematical problem, the number of boundary conditions and their order must be properly related to the order of the differential equations.

The lowest-order theory (the membrane shell theory) leads to a fourth order differential equation, and only four boundary conditions can be satisfied, e.g. we may prescribe only two of the three displacement functions at each end of a cylindrical shell of finite length. This is a well-known property of the membrane theory.

The third-order theory, which is the lowest-order shell theory that includes bending terms, leads to a differential equation of order eight. The eight boundary conditions to a well-posed mathematical problem are those well known from the classical shell theory and will not be repeated here.

Next non-trivial higher-order theory ($n=5$) leads to a differential equation of order ten. This means for example that for a cylindrical shell of finite length, we should prescribe five boundary conditions at each end to have a well-posed mathematical problem.

For any higher-order theory special care has to be taken to describe the boundary conditions properly. There is hardly any doubt associated with the meaning of a 'free' boundary or a 'clamped' boundary, which refer to the absence of stresses or displacements, respectively, throughout the thickness of the shell. But the term 'simply supported' does not have such a well-defined meaning. Since the asymptotic expansion is a true representation of the three-dimensional state, we are in a position to prescribe the boundary conditions precisely as we do in the case of any three-dimensional body, but we can only satisfy them approximately, the approximation depending on the order of our theory.

Let us for example consider a clamped boundary. Strictly speaking a clamped boundary must be understood to mean that all displacements $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ vanish at all points x, y of the boundary throughout the thickness of the shell, i.e. for all $-h \leqslant z \leqslant h$. Thus, all the derivatives $\partial^m/\partial z^m$ of the displacements must vanish for all numbers $m = 0, 1, 2, \ldots$. This can only be satisfied up to an order determined by the order of the differential equation. For the third-order theory we can prescribe the function $U_{(0,0)}$ at the boundary (and similarly for the other two displacements). But since $U_{(0,1)} = -\delta_x W_{(0,0)}$, the derivative $\partial u/\partial z$ can be replaced by the slope $-\partial w/\partial x$. To prescribe the second derivative $\partial^2 u/\partial z^2$ we need a higher-order theory. But since the derivatives with respect to z of any stress or displacement is determined in our analysis in terms of the two-dimensional displacements and their derivatives with respect to x and y, we can accommodate the boundary condition with any accuracy desired, provided that we have a theory of sufficiently high order.

Conditions are very similar for a free boundary. But the exact meaning of for instance a 'simply supported' boundary must be explained in terms of displacements and stresses throughout the thickness of the shell.

For each new iteration the order of the differential equations increases, every new iteration permits (and requires) a more detailed description of the displacements and stresses at the boundary. The so-called three-dimensional boundary layers, which have been discussed to some extent in the literature are a natural result of this. However, it would take us too far to pursue this further.

6. Numerical results

To determine the accuracy of the different order shell theories, it is necessary to have an accurate solution of the three-dimensional cylinder in at least a special but relevant case. For that purpose we have computed the natural frequencies of an infinitely long cylinder of arbitrary thickness vibrating in a sinusoidal pattern in both directions from the three-dimensional equations of equilibrium.[5]

Let us assume that the displacements in the middle surface are given by

[5] The method is described in Appendix II.

$$u = A\cos\frac{px}{R}\cos\frac{qy}{R};\quad v = B\sin\frac{px}{R}\sin\frac{qy}{R};\quad w = C\sin\frac{px}{R}\cos\frac{qy}{R}$$

For a complete shell the displacements must be periodic in y with the period $2\pi R$ and therefore q has to be an integer $\geqslant 0$. The number p determines the wave-length of the deformation in axial direction and cannot vanish, but is otherwise not restricted.

This displacement pattern satisfies all three equations of motion (20)–(22). When substituted into the equations, we get a system of three linear and homogeneous equations for the coefficients A, B and C. The condition for a non-trivial solution (that the determinant of the system vanishes) determines the eigenvalue Λ.

For a numerical evaluation let us take Poisson's ratio $\nu = 1/3$, which makes $\mu = 2$. In order to compare the results with those of the Moreley–Koiter equations we introduce

$$\lambda = \frac{1-\nu^2}{E}\rho R^2\omega^2 = \frac{4(\mu+1)}{\mu+2}\Lambda = 3\Lambda$$

For the case $p = q = 2$ we get the results shown in Fig. 1, where the curve A gives the eigenvalue for $n = 3$. The curve for the eigenvalue according to the Morley–Koiter equations[6] coincides with curve A and within the accuracy of the diagram it cannot be distinguished from the third-order theory. Curve B gives the eigenvalue according to the refined shell theory corresponding to $n = 5$. Curve C is the three-dimensional solution, to which the two-dimensional shell solutions A and B can be compared.

Similar results are shown for a higher mode $p = q = 8$ in Fig. 2, where the curves A, B and C have the same meaning as above in Fig. 1. It is interesting to note that the lower-order theory $n = 3$ gives an upper bound and the higher-order theory $n = 5$ a lower bound for the eigenvalue. We might expect that next higher-order theory $n = 7$ would again yield an upper bound for λ. The same pattern appears, and the same conclusions may be drawn, for all other cases investigated in this connection.

The diagrams illustrate both the improvement obtained by the higher-order theory but certainly also the limitations of any shell theory.

The analysis predicts that in the third-order theory the relative error is proportional to $(h/L)^2$ and in the fifth-order theory proportional to $(h/L)^4$, when the shell is sufficiently thin. This would require the functions

$$A = \left[\frac{\lambda_3}{\lambda} - 1\right] \Big/ \left(\frac{h}{L}\right)^2 \quad \text{and} \quad B = \left[\frac{\lambda_5}{\lambda} - 1\right] \Big/ \left(\frac{h}{L}\right)^4$$

to be independent of h for sufficiently thin shells. Here λ_3 is the eigenvalue according to the third-order theory and λ_5 the eigenvalue according to the fifth-order theory, while λ is the exact eigenvalue.

Fig. 3 shows the functions A and B (in an arbitrary scale, different for the two functions) evaluated using the numerically computed value of λ for the case $p = q = 8$. For curves A1 and B1 at first-order Runge–Kutta method was used to determine λ and for the curves A2 and B2 a higher-order Runge–Kutta was applied.

The result is as one would expect, except for the range $0 < h/R < 0.03$ where clearly the accuracy of the numerical determination of λ is critical and insufficient for this purpose.

[6] Niordson, F.I., 1985 p. 259–262.

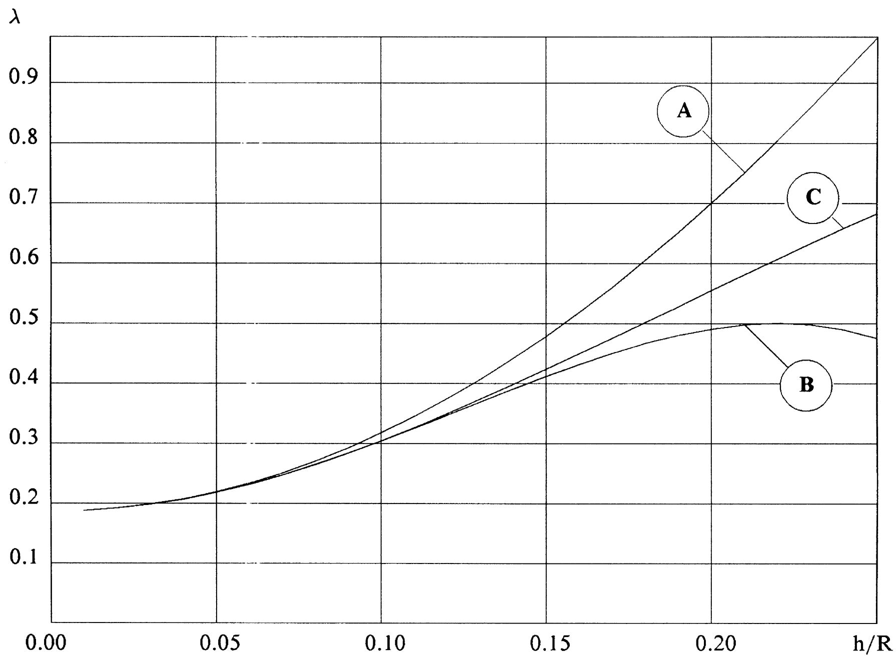

Fig. 1. The eigenvalue λ for the case $p = q = 2$. Curve A: Present third-order theory and Morely–Koiter equations. Curve B: Present fifth-order shell theory. Curve C: Three-dimensional solution.

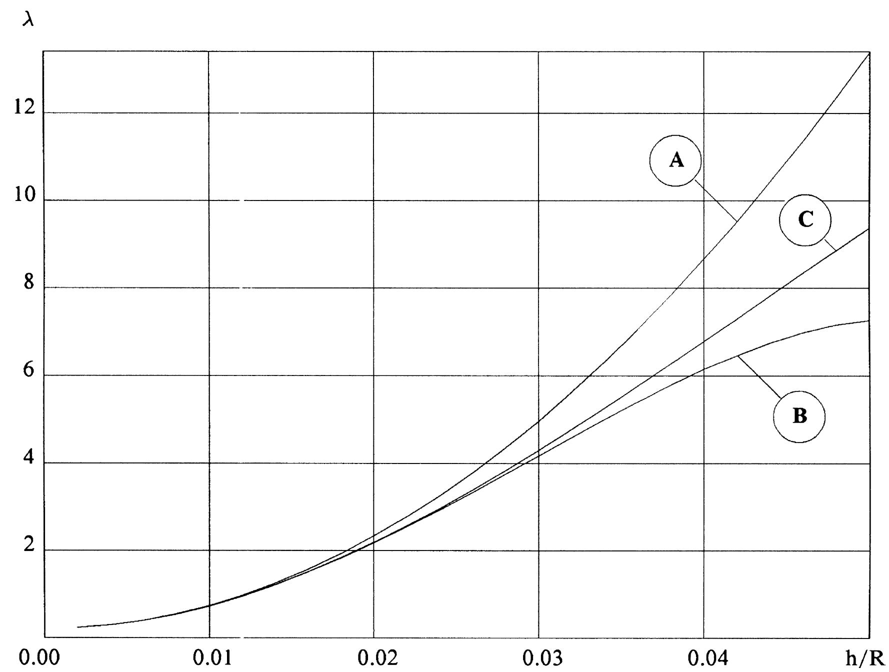

Fig. 2. The eigenvalue λ for the case $p = q = 8$. Curve A: Present third-order theory and Morely–Koiter equations. Curve B: Present fifth-order shell theory. Curve C: Three-dimensional solution.

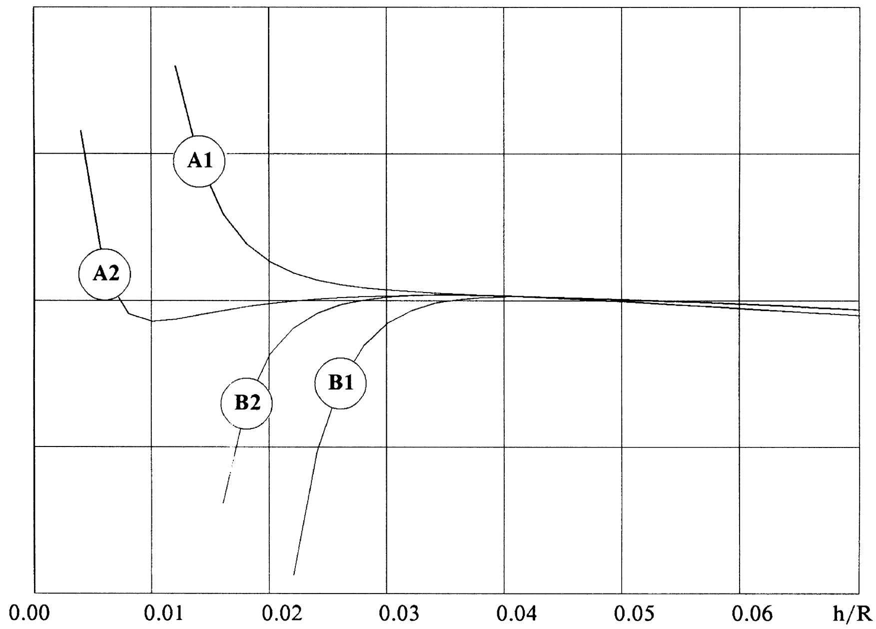

Fig. 3. The function A and B for the case $p = q = 8$. Curve A1: The function A using a first-order Runge–Kutta for λ. Curve B1: The function B using a first-order Runge–Kutta for λ. Curve A2: The function A using a higher-order Runge–Kutta for λ. Curve B2: The function B using a higher-order Runge–Kutta for λ.

Appendix I

List of all non-vanishing functions $f[p, q, r]$, $g[p, q, r]$, $h[p, q, r]$

$$f_{111} = (2 + 3\mu)v(x, y)$$

$$f_{110} = 2\mu w(x, y)/R$$

$$f_{120} = 4(1 + \mu)u(x, y)$$

$$f_{102} = (2 + \mu)u(x, y)$$

$$f_{211} = (2 + 3\mu)u(x, y)$$

$$f_{220} = (2 + \mu)v(x, y)$$

$$f_{201} = 4(1 + \mu)w(x, y)/R$$

$$f_{202} = 4(1 + \mu)v(x, y)$$

$$f_{310} = -2\mu u(x, y)/R$$

$$f_{301} = -4(1 + \mu)v(x, y)/R$$

$$g_{111} = (4 + 3\mu)(-4 - 2\mu + \mu^2)v(x, y)/R^2$$

$$g_{131} = 4\mu^2(1 + \mu)v(x, y)$$

$$g_{112} = 2\left(4 + 10\mu + 12\mu^2 + 5\mu^3\right)w(x, y)/R$$

$$g_{122} = 4\mu^2(1 + \mu)u(x, y)$$

$$g_{113} = 4\mu^2(1 + \mu)v(x, y)$$

$$g_{110} = 4(1 + \mu)(-4 + \mu + 2\mu^2)w(x, y)/R^3$$

$$g_{120} = 2\mu(-4 + \mu + 2\mu^2)u(x, y)/R^2$$

$$g_{130} = 2(1 + \mu)(-8 - 6\mu + \mu^2)w(x, y)/R$$

$$g_{140} = 4\mu^2(1 + \mu)u(x, y)$$

$$g_{102} = (2 + \mu)^3u(x, y)/R^2$$

$$g_{211} = -2(12 + 38\mu + 23\mu^2 + 3\mu^3)u(x, y)/R^2$$

$$g_{221} = 2(-12 - 18\mu - 6\mu^2 + \mu^3)w(x, y)/R$$

$$g_{231} = 4\mu^2(1 + \mu)u(x, y)$$

$$g_{222} = 4\mu^2(1 + \mu)v(x, y)$$

$$g_{213} = 4\mu^2(1 + \mu)u(x, y)$$

$$g_{201} = -2(1 + \mu)(40 + 28\mu + 3\mu^2)w(x, y)/R^3$$

$$g_{202} = -4(1 + \mu)(4 + \mu)(5 + 3\mu)v(x, y)/R^2$$

$$g_{203} = 2\mu(1+\mu)(6+5\mu)w(x, y)/R$$

$$g_{204} = 4\mu^2(1+\mu)v(x, y)$$

$$g_{321} = -\left(16+20\mu+14\mu^2+7\mu^3\right)v(x, y)/R$$

$$g_{312} = -\left(32+76\mu+58\mu^2+17\mu^3\right)u(x, y)/R$$

$$g_{322} = -8(1+\mu)(2+\mu)^2 w(x, y)$$

$$g_{310} = -2\mu(2+\mu)(2+5\mu)u(x, y)/R^3$$

$$g_{320} = -2\left(8+20\mu+17\mu^2+6\mu^3\right)w(x, y)/R^2$$

$$g_{330} = -6\mu^2(1+\mu)u(x, y)/R$$

$$g_{340} = -4(1+\mu)(2+\mu)^2 w(x, y)$$

$$g_{301} = -10\mu(1+\mu)(2+\mu)v(x, y)/R^3$$

$$g_{302} = -4(1+\mu)(20+23\mu+8\mu^2)w(x, y)/R^2$$

$$g_{303} = -6(1+\mu)(8+8\mu+3\mu^2)v(x, y)/R$$

$$g_{304} = -4(1+\mu)(2+\mu)^2 w(x, y)$$

$$h_{111} = -(2816+9800\mu+12216\mu^2+6586\mu^3+1423\mu^4+66\mu^5)v(x, y)/(2R^4)$$

$$h_{131} = -(3784+11972\mu+14562\mu^2+8331\mu^3+1880\mu^4+48\mu^5)v(x, y)/(2R^2)$$

$$h_{151} = 4\mu^2(1+\mu)(-24-28\mu+3\mu^2)v(x, y)$$

$$h_{112} = (-3056-4040\mu+5076\mu^2+12030\mu^3+7505\mu^4+1536\mu^5)w(x, y)/(2R^3)$$

$$h_{122} = (-664-1028\mu+2234\mu^2+5505\mu^3+3868\mu^4+880\mu^5)u(x, y)/(2R^2)$$

$$h_{132} = \left(-1024-2488\mu-1532\mu^2+422\mu^3+735\mu^4+238\mu^5\right)w(x, y)/R$$

$$h_{142} = 8\mu^2(1+\mu)(-24-28\mu+3\mu^2)u(x,y)$$

$$h_{113} = \left(-1000-1440\mu+1368\mu^2+3680\mu^3+2375\mu^4+497\mu^5\right)v(x,y)/R^2$$

$$h_{133} = 8\mu^2(1+\mu)(-24-28\mu+3\mu^2)v(x,y)$$

$$h_{114} = \left(272+1784\mu+3672\mu^2+3338\mu^3+1457\mu^4+272\mu^5\right)w(x,y)/R$$

$$h_{124} = 4\mu^2(1+\mu)(-24-28\mu+3\mu^2)u(x,y)$$

$$h_{115} = 4\mu^2(1+\mu)(-24-28\mu+3\mu^2)v(x,y)$$

$$h_{110} = -(1+\mu)(1728+4336\mu+3170\mu^2+734\mu^3+7\mu^4)w(x,y)/R^5$$

$$h_{120} = -\mu(1728+4336\mu+3170\mu^2+734\mu^3+7\mu^4)w(x,y)/(2R^4)$$

$$h_{130} = (-2704-9040\mu-10956\mu^2-5883\mu^3-1143\mu^4+10\mu^5)w(x,y)/R^3$$

$$h_{140} = \mu(-488-1630\mu-1616\mu^2-427\mu^3+17\mu^4)u(x,y)/R^2$$

$$h_{150} = -2(1+\mu)(648+1488\mu+1114\mu^2+344\mu^3+17\mu^4)w(x,y)/R$$

$$h_{160} = 4\mu^2(1+\mu)(-24-28\mu+3\mu^2)u(x,y)$$

$$h_{102} = 9(2+\mu)^5u(x,y)/R^4$$

$$h_{104} = 24(2+\mu)^5u(x,y)/R^2$$

$$h_{211} = (2880+2256\mu-9976\mu^2-14120\mu^3-6359\mu^4-946\mu^5)u(x,y)/(2R^4)$$

$$h_{221} = -(12896+41136\mu+47192\mu^2+24656\mu^3+5377\mu^4+256\mu^5)w(x,y)/(2R^3)$$

$$h_{231} = (1056+4336\mu+6568\mu^2+4665\mu^3+1801\mu^4+317\mu^5)u(x,y)/R^2$$

$$h_{241} = -(2448+7416\mu+8512\mu^2+4602\mu^3+1139\mu^4+74\mu^5)w(x,y)/R$$

$$h_{251} = 4\mu^2(1+\mu)(-24-28\mu+3\mu^2)u(x,y)$$

$$h_{222} = (-5112 - 10812\mu - 6570\mu^2 + 57\mu^3 + 1536\mu^4 + 448\mu^5)v(x, y)/(2R^2)$$

$$h_{242} = 4\mu^2(1 + \mu)(-24 - 28\mu + 3\mu^2)v(x, y)$$

$$h_{213} = (4008 + 19116\mu + 33862\mu^2 + 28371\mu^3 + 11652\mu^4 + 1896\mu^5)u(x, y)/(2R^2)$$

$$h_{223} = (-3328 - 8776\mu - 8148\mu^2 - 2950\mu^3 - 99\mu^4 + 158\mu^5)w(x, y)/R$$

$$h_{233} = 8\mu^2(1 + \mu)(-24 - 28\mu + 3\mu^2)u(x, y)$$

$$h_{224} = 8\mu^2(1 + \mu)(-24 - 28\mu + 3\mu^2)v(x, y)$$

$$h_{215} = 4\mu^2(1 + \mu)(-24 - 28\mu + 3\mu^2)u(x, y)$$

$$h_{201} = -(1 + \mu)(4944 + 17176\mu + 17720\mu^2 + 7259\mu^3 + 1036\mu^4)w(x, y)/R^5$$

$$h_{202} = -(1 + \mu)(3704 + 13396\mu + 14190\mu^2 + 5974\mu^3 + 881\mu^4)w(x, y)/R^4$$

$$h_{203} = (1 + \mu)(-1736 + 1664\mu + 6386\mu^2 + 4645\mu^3 + 1042\mu^4)w(x, y)/R^3$$

$$h_{204} = (1 + \mu)(384 + 5924\mu + 8976\mu^2 + 5010\mu^3 + 977\mu^4)w(x, y)/R^2$$

$$h_{205} = 4(1 + \mu)(-220 - 120\mu + 211\mu^2 + 202\mu^3 + 58\mu^4)w(x, y)/R$$

$$h_{206} = 4\mu^2(1 + \mu)(-24 - 28\mu + 3\mu^2)v(x, y)$$

$$h_{321} = -(2224 + 10160\mu + 15708\mu^2 + 12600\mu^3 + 6256\mu^4 + 1321\mu^5)v(x, y)/(4R^3)$$

$$h_{341} = -(3048 + 10028\mu + 12290\mu^2 + 7023\mu^3 + 2039\mu^4 + 341\mu^5)v(x, y)/(2R)$$

$$h_{312} = (720 - 9664\mu - 40356\mu^2 - 48568\mu^3 - 24300\mu^4 - 4371\mu^5)u(x, y)/(4R^3)$$

$$h_{322} = -(6344 + 23976\mu + 34860\mu^2 + 24697\mu^3 + 8707\mu^4 + 1309\mu^5)w(x, y)/R^2$$

$$h_{332} = (8 - 3020\mu - 8198\mu^2 - 8073\mu^3 - 3761\mu^4 - 843\mu^5)u(x, y)/(2R)$$

$$h_{342} = -12(1 + \mu)(2 + \mu)^3(34 + 27\mu)w(x, y)$$

$$h_{323} = -(6088 + 20564\mu + 27434\mu^2 + 18231\mu^3 + 6359\mu^4 + 1101\mu^5)v(x, y)/(2R)$$

$$h_{314} = (8 - 1764\mu - 5526\mu^2 - 6129\mu^3 - 3021\mu^4 - 631\mu^5)u(x, y)/(2R)$$

$$h_{324} = -12(1 + \mu)(2 + \mu)^3(34 + 27\mu)w(x, y)$$

$$h_{310} = 2\mu(2 + \mu)(120 + 4\mu - 262\mu^2 - 101\mu^3)u(x, y)/R^5$$

$$h_{320} = -(1536 + 6880\mu + 12728\mu^2 + 11550\mu^3 + 5550\mu^4 + 1069\mu^5)w(x, y)/(2R^4)$$

$$h_{330} = -(\mu(324 + 422\mu + 551\mu^2 + 732\mu^3 + 247\mu^4)u(x, y)/R^3)$$

$$h_{340} = -(1536 + 6120\mu + 9262\mu^2 + 6636\mu^3 + 2313\mu^4 + 357\mu^5)w(x, y)/R^2$$

$$h_{350} = -2\mu(1 + \mu)(314 + 354\mu + 132\mu^2 + 53\mu^3)u(x, y)/R$$

$$h_{360} = -4(1 + \mu)(2 + \mu)^3(34 + 27\mu)w(x, y)$$

$$h_{301} = (1 + \mu)(2 + \mu)(356 + 120\mu - 591\mu^2 - 202\mu^3)v(x, y)/R^5$$

$$h_{302} = -(1 + \mu)(1704 + 13796\mu + 19602\mu^2 + 10578\mu^3 + 1979\mu^4)w(x, y)/R^4$$

$$h_{303} = -(1 + \mu)(752 + 8552\mu + 12424\mu^2 + 6767\mu^3 + 1272\mu^4)v(x, y)/R^3$$

$$h_{304} = -(1 + \mu)(3808 + 11404\mu + 12096\mu^2 + 5514\mu^3 + 993\mu^4)w(x, y)/R^2$$

$$h_{305} = -4(1 + \mu)(380 + 937\mu + 956\mu^2 + 445\mu^3 + 95\mu^4)v(x, y)/R$$

$$h_{306} = -4(1 + \mu)(2 + \mu)^3(34 + 27\mu)w(x, y)$$

Appendix II

The three-dimensional equations of motion for circular cylinder are given by (5)–(7). In the numerical evaluation we take $\mu = 2$ and $R = 1$ for simplicity. Substituting the displacement functions

$$u = U(z)\cos(px)\cos(qy)$$

$$v = V(z)\sin(px)\sin(qy)$$

$$w = W(z)\sin(px)\cos(qy)$$

into the equations of motion, we get after cancelling the common trigonometric factor, the following system of ordinary second-order differential equations

$$U''(z) + \frac{1}{1+z}U'(z) - \frac{4p^2+q^2}{(1+z)^2}U(z) + 3pqV(z) + 3pW'(z) + \frac{3p}{1+z}W(z) + \Lambda U(z) = 0$$

$$V''(z) + \frac{3}{1+z}V'(z) - \left(\frac{4q^2}{(1+z)^2} + p^2\right)V(z) + \frac{3pq}{(1+z)^2}U(z) - \frac{3q}{(1+z)^2}W'(z) - \frac{5q}{(1+z)^3}W(z) + \Lambda V(z) = 0$$

$$4W''(z) + \frac{4}{1+z}W'(z) - \left(\frac{4+q^2}{(1+z)^2} + p^2\right)W(z) + 3qV'(z) - \frac{2q}{1+z}V(z) - 3pU'(z) + \Lambda W(z) = 0$$

This system of equations is solved numerically by a Runge–Kutta procedure from $z=-h$ to $z=+h$. The initial values of the first derivatives are determined from the condition that the stresses vanish at $z=-h$, i.e.

$$U'(-h) = -pW(-h)$$

$$V'(-h) = \frac{q}{(1-h)^2}W(-h)$$

$$W'(-h) = \frac{p}{2}U(-h) - \frac{q}{2}V(-h) - \frac{1}{2(1-h)}W(-h)$$

At the upper limit $z=+h$ the stresses are found from

$$\Sigma^{31} = U'(h) + pW(h)$$

$$\Sigma^{32} = V'(h) - \frac{q}{(1+h)^2}W(h)$$

$$\Sigma^{33} = 4W'(h) + \frac{2}{1+h}W(h) + 2qV(h) - 2pU(h)$$

For a given value of Λ we perform the Runge–Kutta integration three times, one for initial values $U(-h)=1$; $V(-h)=0$; $W(-h)=0$ another for the initial values $U(-h)=0$; $V(-h)=1$; $W(-h)=0$ and a third for $U(-h)=0$; $V(-h)=0$; $W(-h)=1$.

The determinant of the three stresses for the three cases is now found as a function of Λ and then the eigenvalue Λ is found from the condition that the determinant vanishes.

References

Brod, K., 1972. Herleitung der Plattengleichung der klassichen Elastizitätstheore durch systematische Entwicklung nach einem Dickenparameter. Diplomarbeit, Göttingen.

Koiter, W.T., 1960. A consistent first approximation in the theory of thin elastic shells. In: Proc. of the IUTAM Symp. On the Theory of Thin Elastic Shells, Amsterdam, 12–33.

Koiter, W.T., 1968. Summary of equations for modified, simplest possible accurate linear theory of thin circular cylindrical shells. Report 442, Lab. Techn. Mech. T.H. Delft.

Morley, L.S.D., 1959. An improvement on Donnell's approximation for thin-walled circular cylinders. Quart. J. Mech. Appl. Math. 12, 89.

Niordson, F., 1971. A note on the strain energy of elastic shells. Int. J. Solids Structures 7, 1570–1573.

Niordson, F., 1979. An asymptotic theory for vibrating plates. Int. J. Solids Structures 15, 167–181.

Niordson, F., 1985. Shell Theory. North-Holland Series Appl. Math. Mech.

Novozhilov, V.V., Finkel'shtein, R.M., 1943. Regarding the errors connected with the Kirchhoff hypotheses in the theory of shells. Prikl. Mat. Mekh. Vol. VII.

Simmonds, J.G., 1997. Some comments on the status of shell theory at the end of the 20th century: complaints and correctives. American Inst. Aeronautics Astronautics, pp. 1–10.